JOURNAL OF APPROXIMATION THEORY **85**, 201–217 (1996)
ARTICLE NO. 0037

Simultaneous Approximation and Quasi-Interpolants

KANG ZHAO

Department of Mathematics, University of Utah, Salt Lake City, Utah 84112, U.S.A

Communicated by Rong-Qing Jia

Received October 11, 1994

It is shown that the shift-invariant space $S(\varphi)$ generated by $\varphi \in W_2^m(\mathbb{R}^s)$ provides simultaneous approximation order k, with $k > m$, iff $S(D^\alpha \varphi)$ provides approximation order $k-m$ in the $L_2(\mathbb{R}^s)$-norm to all functions in $D^\alpha W_2^k(\mathbb{R}^s)$ for each $|\alpha| = m$. Without appealing to the argument of polynomial reproduction, an explicit formula is presented for construction of quasi-interpolant of semi-discrete convolution type that achieves the approximation order provided by $S(\varphi)$. The traditional condition that the symbol $\sum_{\alpha \in \mathbb{Z}^s} \varphi(\alpha)\, e_\alpha$ does not vanish on $[-\pi \cdots \pi]^s$ is reduced to that it does not vanish on a neighborhood of the origin. © 1996 Academic Press, Inc.

1. INTRODUCTION

The paper might be viewed as a continuation and an application of [3] and [17]. de Boor *et al.* [3] have explored some fundamental properties of shift-invariant spaces generated by a single function in $L_2(\mathbb{R}^s)$. By definition, a linear space S consisting of functions defined on $\mathbb{R}^s$ is said to be **shift invariant** when S is invariant under translation by integer points. For φ in $L_2(\mathbb{R}^s)$, denote by $S(\varphi)$ the closure of the linear span of the shifts $\varphi(\cdot - \alpha)$ with $\alpha \in \mathbb{Z}^s$. $S(\varphi)$ is called the **principal shift-invariant space** generated by φ. A characterization of $S(\varphi)$ obtained by [3] is that $f \in L_2(\mathbb{R}^s)$ lies in $S(\varphi)$ iff there exists a 2π-periodic function τ such that $\hat{f} = \hat{\varphi}\tau$, where $\hat{f}$ denotes the Fourier transform of f. As to the approximation order, [3] proved that $S(\varphi)$ provides approximation order k iff there exists a constant C such that

$$\Lambda_\varphi := \left(1 - \frac{|\hat{\varphi}|^2}{[\hat{\varphi}, \hat{\varphi}]}\right)^{1/2} \leqslant C\, |\cdot|^k \tag{1.1}$$

holds almost everywhere (a.e.) on a neighborhood of the origin, where $[\hat{\varphi}, \hat{\varphi}] := \sum_{\alpha \in \mathbb{Z}^s} |\hat{\varphi}(\cdot - 2\pi\alpha)|^2$. Here and below we adopt the *convention* that a fraction is zero when its numerator is zero, even if its denominator

0021-9045/96 $18.00
Copyright © 1996 by Academic Press, Inc.
All rights of reproduction in any form reserved.

is also zero. After their work, [17] obtained similar results on shift-invariant subspaces generated by one element in a product space of a finite number of $L_2(\mathbb{R}^s)$. In particular, it has been proved by [17] that $S(\varphi)$ provides **simultaneous approximation order** (m, k) iff there exists a constant C such that

$$\Lambda_{\varphi, m} := \left(1 - \frac{(1 + |\cdot|^{2m})\, |\hat{\varphi}|^2}{[\hat{\varphi}, \hat{\varphi}] + [\hat{\varphi}, \hat{\varphi}]_m}\right)^{1/2} \leqslant C\, |\cdot|^k \tag{1.2}$$

holds a.e. on a neighborhood of the origin, where $[\hat{\varphi}, \hat{\varphi}]_m := [|\cdot|^m\, \hat{\varphi}, |\cdot|^m\, \hat{\varphi}]$. In other words, there is a constant C such that, for $h \to 0+$,

$$\inf_{g \in S(\varphi)} \sum_{j=0}^{m} h^j\, |g(\cdot/h) - f|_{j,2} \leqslant Ch^k(|f|_{k,2} + |f|_{m,2}), \qquad \forall f \in W_p^k(\mathbb{R}^s) \cap W_p^m(\mathbb{R}^s)$$

iff (1.2) holds a.e. on a neighborhood of the origin, where $|\cdot|_{j,2}$ is the Sobolev seminorm:

$$|f|_{j,2} := \left(\int_{\mathbb{R}^s} |\cdot|^{2j}\, |\hat{f}|^2\right)^{1/2}.$$

When j is a nonnegative integer, from the definition we know that $|\cdot|_{j,2}$ is equivalent to the seminorm defined by

$$\left(\sum_{|\alpha| = j} \|D^\alpha f\|_2^2)^{1/2}\right)^{1/2}, \qquad \forall f \in W_2^m(\mathbb{R}^s).$$

It is clear that $S(\varphi)$ providing simultaneous approximation order $(0, k)$ is equivalent to $S(\varphi)$ providing approximation order k (in the $L_2(\mathbb{R}^s)$-norm).

As we shall see in the next section, an equivalent characterization to (1.2) is that

$$\sum_{\alpha \in \mathbb{Z}^s \backslash 0} |\hat{\varphi}(\cdot - 2\pi\alpha)|^2\, |\cdot - 2\pi\alpha|^{2m} \leqslant C_1\, |\cdot|^{2k}\, |\hat{\varphi}|^2 \tag{1.3}$$

holds a.e. on a neighborhood of the origin, for some constant C_1. With aid of (1.3) we shall show some relation between simultaneous approximation and approximation in the Sobolev seminorm $|\cdot|_{m,2}$. That is, $S(\varphi)$ provides simultaneous approximation order (m, k) with $k > m$, iff it provides approximation order $k - m$ in the Sobolev seminorm $|\cdot|_{m,2}$, which is well known in the setting of univariate cardinal spline. This result is of interest because there is no decay condition on the generator φ. Moreover, it follows a corollary that if $S(D^\alpha\varphi)$ provides approximation order $k > 0$ in the $L_2(\mathbb{R}^s)$-norm to all functions in $D^\alpha W_2^{k+m}(\mathbb{R}^s)$ for each $|\alpha| = m$ then $S(\varphi)$ provides simultaneous approximation order $(m, k+m)$, where D^α is the α-order differential operator. This corollary is of interest in studies of piecewise polynomial function spaces because differential operators reduce the degree of polynomials and the smoothness of piecewise polynomial functions. As we know, if a space S contains all continuous piecewise polynomial functions of degree k for some "triangulation" of $\mathbb{R}^s$ then S provides approximation order $\geqslant k+1$.

In Section 3, we consider approximation via quasi-interpolants of semi-discrete convolution type. We note that there are many papers working on quasi-interpolants with a compactly supported generator φ or with a generator φ satisfying certain decay conditions at the infinity, among which are [1, 8, 10, 12, 14]. As shown by [12], one can construct a quasi-interpolant with φ that achieves approximation order k if φ satisfies the decay condition

$$\sum_{\alpha \in \mathbb{Z}^s} |\cdot - \alpha|^k \, |\varphi(\cdot - \alpha)| \in L_p([0..1]^s) \tag{1.4}$$

and the **Strang-Fix conditions of order** k: $\hat{\varphi}(0) \neq 0$ and $D^\alpha \hat{\varphi}$ vanishes on $2\pi\mathbb{Z}^s \backslash 0$ for $|\alpha| < k$. Our approach is novel for construction of quasi-interpolants because we do not appeal to the argument of polynomial reproduction used in the literature. In contrast to the argument of polynomial reproduction, we give an explicit formula for the construction. More specifically, for any $\varphi \in W_2^m(\mathbb{R}^s)$ with $m > s/2$, if $S(\varphi)$ provides simultaneous approximation order (m, k) and the reciprocal of

$$\tilde{\varphi} := \sum_{\alpha \in \mathbb{Z}^s} \varphi(\alpha) \, e_\alpha$$

is essentially bounded on a neighborhood B of the origin then there is a sequence $b \in l_2(\mathbb{Z}^s)$ such that

$$\left\| \sum_{\alpha \in \mathbb{Z}^s} \varphi(\cdot/h - \alpha) \sum_{\beta \in \mathbb{Z}^s} b(\alpha - \beta) \, f(h\beta) - f \right\|_2 = O(h^k), \qquad \forall f \in W_2^k(\mathbb{R}^s), \tag{1.5}$$

where $e_\alpha \colon x \mapsto e^{ix\alpha}$ and b consists of the Fourier series coefficients of $\chi_B/\tilde{\varphi}$, with χ_B the characteristic function of B. Clearly, this is a generalization of the known result for the case φ is compactly supported and $\tilde{\varphi}$ does not vanish on $[-\pi \cdots \pi]^s$. For such a generator φ, [2, 10] showed via the argument of polynomial reproduction that $S(\varphi)$ provides approximation order k if φ satisfies the Strang-Fix conditions of order k. Since the quasi-interpolant in (1.5) needs the function values of f at the scaled lattice $h\mathbb{Z}^s$ only, it relates itself to the theory of cardinal interpolation.

2. Approximation in Sobolev Seminorm

Let φ be a compactly supported function in $W_p^m(\mathbb{R}^s)$. As shown by Zhao [18], if the shift-invariant space generated by φ provides approximation order k in the $L_p(\mathbb{R}^s)$-norm, then the space automatically provides simultaneous approximation order (m, k). An interesting question regarding $S(\varphi)$ is: Will $S(\varphi)$ provide approximation order k if $S(D^\alpha\varphi)$ provides approximation order $k-m>0$ in the $L_p(\mathbb{R}^s)$-norm to all functions in $D^\alpha W_p^k(\mathbb{R}^s)$ for each $|\alpha|=m$? When φ is a compactly supported function with $\hat{\varphi}(0)\neq 0$, the answer to the question is essentially known. When $\hat{\varphi}(0)\neq 0$, we know that $S(\varphi)$ provides approximation order k iff $S(\varphi)$ locally contains all polynomials of degree $<k$ [11, 14]. When $S(D^\alpha\varphi)$ provides approximation order $k-m>0$ in the $L_p(\mathbb{R}^s)$-norm to all functions in $D^\alpha W_p^k(\mathbb{R}^s)$ for every $\alpha\in\mathbb{Z}_+^s$ satisfying that $|\alpha|=m$, the argument of Theorem 5 in [11] shows that $S(D^\alpha\varphi)$ locally contains all polynomials of degree $<k-m$, for $\alpha\in\mathbb{Z}_+^s$ satisfying that $|\alpha|=m$. Therefore, $D^\alpha S(\varphi)$ locally contains all polynomials of degree $<k-m$. It follows that $S(\varphi)$ locally contains all polynomials of degree $<k$ because $S(\varphi)$ is shift invariant. Consequently, $S(\varphi)$ provides approximation order k.

When $p=2$, we shall see in this section that we still have a confirmative answer for the above question, for *any* function $\varphi\in W_2^k(\mathbb{R}^s)$. Due to [3], principal shift-invariant subspaces of $L_2(\mathbb{R}^s)$ become known much more than those in $L_p(\mathbb{R}^s)$ with p other than 2. Of the most important result is the characterization of the orthogonal projector to $S(\varphi)$. Following [3, 17] obtained a characterization of the projector to a shift-invariant subspace of any product space of $L_2(\mathbb{R}^s)$. By Theorem 2.2 in [17] the following result is immediate.

Theorem 2.1. *Let φ be any function in $W_2^m(\mathbb{R}^s)$. Then $g\in S(\varphi)$ is the best approximation to $f\in W_2^m(\mathbb{R}^s)$ in the seminorm $|\cdot|_{m,2}$ iff g satisfies that $\hat{g}=([\hat{f},\hat{\varphi}]_m/[\hat{\varphi},\hat{\varphi}]_m)\,\hat{\varphi}$, where*

$$[\hat{f},\hat{\varphi}]_m := \sum_{\alpha\in\mathbb{Z}^s} \hat{f}(\cdot-2\pi\alpha)\,\overline{\hat{\varphi}(\cdot-2\pi\alpha)}\,|\cdot-2\pi\alpha|^{2m}.$$

For f and g in $W_2^m(\mathbb{R}^s)$, $|f-g(\cdot/h)|_{m,2}=h^{s/2-m}\,|f_h-g|_{m,2}$, where $f_h:=f(h\cdot)$. Therefore,

$$|f-g(\cdot/h)|_{m,2}=O(h^k)\Leftrightarrow\int_{\mathbb{R}^s}|\cdot|^{2m}\,|\widehat{f_h}-\hat{g}|^2=O(h^{2(k+m)-s}).$$

Since $|hx|>1$ for $x\in(\mathbb{R}^s\backslash[-\pi..\pi]^s)/h$,

$$h^{2m-s}\int_{(\mathbb{R}^s\backslash[-\pi..\pi]^s)/h}|\cdot|^{2m}\,|\hat{f}|^2\leqslant h^{2(k+m)-s}\int_{(\mathbb{R}^s\backslash[-\pi..\pi]^s)/h}|\cdot|^{2(k+m)}\,|\hat{f}|^2.$$

This proves that, for $f \in W_2^{k+m}(\mathbb{R}^s)$,

$$\int_{\mathbb{R}^s} (1-\chi_{[-\pi..\pi]^s})\, |\cdot|^{2m}\, |\widehat{f_h}|^2 = \varepsilon_f^2(h)\, |f|_{k+m,2}^2\, h^{2(k+m)-s}, \tag{2.1}$$

with $\varepsilon_f(h) \geqslant 0$ defined by

$$\varepsilon_f^2(h) := \frac{\int_{(\mathbb{R}^s \backslash [-\pi..\pi]^s)/h} |\cdot|^{2(k+m)}\, |\hat{f}|^2}{|f|_{k+m,2}^2}$$

goes to 0 as $h \to 0+$.

For any $f, g \in W_2^m(\mathbb{R}^s)$, without loss of generality, assume that

$$E(f, S(\varphi))_{m,2} := \inf_{q \in S(\varphi)} |f-q|_{m,2} \geqslant E(g, S(\varphi))_{m,2}.$$

Suppose that ψ and ϕ are the functions in $S(\varphi)$ such that $E(f-g, S(\varphi))_{m,2} = |(f-g)-\psi|_{m,2}$ and $E(g, S((\varphi))_{m,2} = |g-\varphi|_{m,2}$. Then,

$$0 \leqslant |f-(\phi+\psi)|_{m,2} - |g-\phi|_{m,2} \leqslant |(f-g)-\psi|_{m,2}.$$

Thus we obtain

$$|E(f, S(\varphi))_{m,2} - E(g, S(\varphi))_{m,2}| \leqslant E(f-g, S(\varphi))_{m,2}. \tag{2.2}$$

Denote by $f^{\vee}$ the **inverse Fourier transform** of f and denote by $\chi_{[-\pi..\pi]^s}$ the characteristic function of $[-\pi..\pi]^s$. From (2.1) and (2.2) we obtain

$$|E(f_h, S(\varphi))_{m,2} - E((\chi_{[-\pi..\pi]^s}\widehat{f_h})^{\vee}, S(\varphi))_{m,2}| \leqslant \varepsilon_f(h)\, |f|_{k+m,2}\, h^{k+m-s/2}.$$

Consequently, $S(\varphi)$ provides approximation order k in the seminorm $|\cdot|_{m,2}$ iff there exists constant C such that

$$E((\chi_{[-\pi..\pi]^s}\widehat{f_h})^{\vee}, S(\varphi))_{m,2} \leqslant C\, |f|_{k+m,2}\, h^{k+m-s/2}, \quad \forall f \in W_2^{k+m}(\mathbb{R}^s). \tag{2.3}$$

THEOREM 2.2. *For $\varphi \in W_2^m(\mathbb{R}^s)$, $S(\varphi)$ provides approximation order k in the seminorm $|\cdot|_{m,2}$ iff $|\cdot|^{-k}\, \Lambda_m$ is essentially bounded on a neighborhood of the origin, where*

$$\Lambda_m := \left(1 - \frac{|\cdot|^{2m}\, |\hat{\varphi}|^2}{[\hat{\varphi}, \hat{\varphi}]_m}\right)^{1/2}.$$

Proof. It suffices to show (2.3) holds for $f \in L_2(\mathbb{R}^s)$ such that $\widehat{f_h}$ is supported on $[-\pi..\pi]^s$. Let g be the best approximation from $S(\varphi)$ to f_h in the seminorm $|\cdot|_{m,2}$. Then, by Theorem 2.1,

$$\hat{g} = \frac{[\widehat{f_h}, \hat{\varphi}]_m}{[\hat{\varphi}, \hat{\varphi}]_m} \hat{\varphi}.$$

By the orthogonality,

$$\int_{\mathbb{R}^s} |\cdot|^{2m} |\widehat{f_h} - \hat{g}|^2 = \int_{[-\pi..\pi]^s} |\cdot|^{2m} |\widehat{f_h}|^2 - \int_{\mathbb{R}^s} |\cdot|^{2m} |\hat{g}|^2.$$

Since $\widehat{f_h}$ is supported on $[-\pi..\pi]^s$, on $2\pi\alpha + [-\pi..\pi]^s$,

$$|\cdot|^{2m} |\hat{g}|^s = \frac{|\cdot - 2\pi\alpha|^{4m} |\widehat{f_h}(\cdot - 2\pi\alpha)\, \hat{\varphi}(\cdot - 2\pi\alpha)|^s}{[\hat{\varphi}, \hat{\varphi}]_m^2} |\cdot|^{2m} |\hat{\varphi}|^2.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^s} |\cdot|^{2m} |\hat{g}|^2 &= \int_{[-\pi..\pi]^s} |\cdot|^{2m} |\widehat{f_h}|^2 \frac{|\cdot|^{2m} |\hat{\varphi}|^2}{[\hat{\varphi}, \hat{\varphi}]_m^2} [\hat{\varphi}, \hat{\varphi}]_m \\ &= \int_{[-\pi..\pi]^s} |\cdot|^{2m} |\widehat{f_h}|^2 \frac{|\cdot|^{2m} |\hat{\varphi}|^2}{[\hat{\varphi}, \hat{\varphi}]_m}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}\int_{\mathbb{R}^s} |\cdot|^{2m} |\widehat{f_h} - \hat{g}|^2 &= \int_{[-\pi..\pi]^s} |\cdot|^{2m} |\widehat{f_h}|^2 \left(1 - \frac{|\cdot|^{2m} |\, |\hat{\varphi}|^2}{[\hat{\varphi}, \hat{\varphi}]_m}\right) \\ &= \int_{\mathbb{R}^s} |\cdot|^{2m} |\widehat{f_h}|^2 \Lambda_m^2.\end{aligned}$$

Since $\widehat{f_h} = h^{-s} \hat{f}(\cdot/h)$,

$$\begin{aligned}\int_{[-\pi..\pi]^s} |\cdot|^{2m} |\widehat{f_h}|^2 \Lambda_m^2 &= h^{-2s} \int_{[-\pi..\pi]^s} |\cdot|^{2m} |\hat{f}(\cdot/h)|^2 \Lambda_m^2 \\ &= h^{-s} \int_{[-\pi..\pi]^s/h} |h\cdot|^{2m} |\hat{f}|^2 \Lambda_m^2(h\cdot) \\ &= h^{2(k+m)-s} \int_{[-\pi..\pi]^s/h} |\cdot|^{2(k+m)} |\hat{f}|^2 \frac{\Lambda_m^2(h\cdot)}{|h\cdot|^{2k}}.\end{aligned}$$

This proves that (2.3) holds iff $\Lambda_m^2(h\cdot) \leqslant \text{const}\, |h\cdot|^{2k}$ holds a.e. on $[-\pi..\pi]^s/h$. Equivalently, $S(\varphi)$ provides approximation order k in the seminorm $|\cdot|_{m,2}$ iff $|\cdot|^{-k} \Lambda_m$ is essentially bounded on $[-\pi..\pi]^s$.

Replacing $[-\pi..\pi]^s$ by any neighborhood of the origin in the above argument, we can show that $S(\varphi)$ provides approximation order k in the seminorm $|\cdot|_{m,2}$ iff $|\cdot|^{-k} \Lambda_m$ is essentially bounded on any neighborhood of the origin. ∎

LEMMA 2.3. *Let f and g be two nonnegative functions and k a positive number. Then $|\cdot|^{-k} f/(g+f)$ is essentially bounded on a neighborhood of the origin iff $(|\cdot|^k g)^{-1} f$ is essentially bounded on a neighborhood of the origin.*

Proof. When $|\cdot|^{-k} f/(g+f)$ is essentially bounded on some neighborhood Ω of the origin, there exists a constant C such that

$$f \leqslant C\,|\cdot|^k\,(g+f).$$

Thus we have that $(1-C\,|\cdot|^k) f \leqslant C\,|\cdot|^k\,g$. Thus, for almost every x in Ω satisfying that $C\,|x|^k < 1/2$,

$$f(x) \leqslant 2C\,|x|^k\,g(x). \tag{2.4}$$

This proves that $(|\cdot|^k g)^{-1} f$ is essentially bounded on some neighborhood of the origin. Conversely, if (2.4) holds for some constant C a.e. on a neighborhood Ω of the origin, then

$$\frac{f}{g+f} \leqslant \frac{f}{g} \leqslant 2C\,|\cdot|^k. \tag{2.5}$$

It is clear that (2.5) implies that $|\cdot|^{-k} f/(g+f)$ is essentially bounded on Ω. ∎

From Theorem 2.2 and Lemma 2.3 we obtain

COROLLARY 2.4. *For $\varphi \in W_2^m(\mathbb{R}^s)$, $S(\varphi)$ provides approximation order $k>0$ in the seminorm $|\cdot|_{m,2}$ iff there exists a constant C and a neighborhood of the origin on which*

$$\sum_{a \in \mathbb{Z}^s \backslash 0} |\hat{\varphi}(\cdot - 2\pi\alpha)|^2\,|\cdot - 2\pi\alpha|^{2m} \leqslant C\,|\cdot|^{2(k+m)}\,|\hat{\varphi}|^2 \tag{2.6}$$

holds almost everywhere.

It has been proved by Zhao [17] that, for $\varphi \in W_2^m(\mathbb{R}^s)$, $S(\varphi)$ provides simultaneous approximation order (m, k) if and only if there exist a constant C and a neighborhood Ω of the origin such that

$$\Lambda_{\varphi,m} = \left(1 - \frac{(1+|\cdot|^{2m})\,|\hat{\varphi}|^2}{[\hat{\varphi}, \hat{\varphi}] + [\hat{\varphi}, \hat{\varphi}]_m}\right)^{1/2} \leqslant C\,|\cdot|^k \tag{2.7}$$

holds a.e. on Ω. From this result and Lemma 2.3 we obtain

COROLLARY 2.5. *For any* $\varphi \in W_2^m(\mathbb{R}^s)$, $S(\varphi)$ *provides simultaneous approximation order* (m, k), *with* $k > 0$, *iff there exist a constant* C *and a neighborhood of the origin on which*

$$\sum_{\alpha \in \mathbb{Z}^s \backslash 0} |\hat{\varphi}(\cdot - 2\pi\alpha)|^2 \, |\cdot - 2\pi\alpha|^{2m} \leqslant C \, |\cdot|^{2k} \, |\hat{\varphi}|^2 \tag{2.8}$$

holds almost everywhere.

Proof. By [17], $S(\varphi)$ provides simultaneous approximation order (m, k) iff (2.7) holds a.e. on a neighborhood of the origin, for some constant C. By Lemma 2.3, this is equivalent to that

$$\sum_{\alpha \in \mathbb{Z}^s \backslash 0} |\hat{\varphi}(\cdot - 2\pi\alpha)|^2 \, (1 + |\cdot - 2\pi\alpha|^{2m}) \leqslant C_1 \, |\cdot|^{2k} \, (1 + |\cdot|^{2m}) \, |\hat{\varphi}|^2 \tag{2.9}$$

holds a.e. on a neighborhood of the origin. Note that $|x - 2\pi\alpha| > 1$ for $|x| < 1$ and $\alpha \in \mathbb{Z}^s \backslash 0$. Hence, (2.9) holds a.e. on a neighborhood of the origin iff (2.8) holds a.e. on a neighborhood of the origin. ▮

We note that Corollary 2.5 also gives a modified version for the characterization obtained by [3] that $|\cdot|^{-k} \Lambda_\varphi$ is essentially bounded on a neighborhood of the origin, corresponding to the case $m = 0$. Combining Corollary 2.4 and Corollary 2.5 yields

THEOREM 2.6. *Let* φ *be a function in* $W_2^m(\mathbb{R}^s)$ *and* k *a number* $> m$. $S(\varphi)$ *provides simultaneous approximation order* (m, k) *iff* $S(\varphi)$ *provides approximation order* $k - m$ *in the seminorm* $|\cdot|_{m, 2}$.

Note that if $S(\varphi)$ provides simultaneous approximation order (m, k) then it provides approximation order k in the $L_2(\mathbb{R}^s)$-norm.

COROLLARY 2.7 *For any* $\varphi \in W_2^m(\mathbb{R}^s)$, *if* $S(\varphi)$ *provides approximation order* $k > 0$ *in the seminorm* $|\cdot|_{m, 2}$ *then it provides approximation order* $k + m$ *in the* $L_2(\mathbb{R}^s)$*-norm.*

As proved by [18] if $\varphi \in W_2^m(\mathbb{R}^s)$ is compactly supported and $S(\varphi)$ provides approximation order $k > m$ in the $L_2(\mathbb{R}^s)$-norm then it provides simultaneous approximation order (m, k). Thus we have

COROLLARY 2.8 *For any compactly supported function* $\varphi \in W_2^m(\mathbb{R}^s)$, $S(\varphi)$ *provides approximation order* $k + m$, *with* $k > 0$, *iff* $S(\varphi)$ *provides approximation order* k *in the seminorm* $|\cdot|_{m, 2}$.

We next show that the simultaneous approximation is automatically provided by principal shift-invariant spaces in the sense that if $S(D^\alpha \varphi)$ provides approximation order $k > 0$ to all functions in $D^\alpha W_2^{k+m}(\mathbb{R}^s)$ for each $|\alpha| = m$ then $S(\varphi)$ provides simultaneous approximation order $(m, k+m)$. Recall that $f_h = f(\cdot/h)$. It is clear that

$$\|D^\alpha(f - g(\cdot/h))\|_2 = \|D^\alpha f - h^{-m}(D^\alpha g)(\cdot/h)\|_2 = h^{s/2-m} \|D^\alpha(f_h - g)\|_2,$$

By the characterization of orthogonal projection obtained by [4], the best approximation from $S(D^\alpha\varphi)$ to $D^\alpha f_h$ is the inverse Fourier transform of

$$\frac{[\widehat{f_h}, \hat{\varphi}]_\alpha}{[\hat{\varphi}, \hat{\varphi}]_\alpha} (i\cdot)^\alpha \hat{\varphi}, \tag{2.10}$$

where

$$[\hat{f}, \hat{\varphi}]_\alpha := \sum_{\beta \in \mathbb{Z}^s} (\cdot - 2\pi\beta)^{2\alpha} \hat{f}(\cdot - 2\pi\beta) \overline{\hat{\varphi}(\cdot - 2\pi\beta)}.$$

From the characterization of principal shift-invariant space obtained by [4] we know that f lies in $S(D^\alpha\varphi)$ iff there exists a 2π-periodic function τ such that

$$\hat{f} = (i\cdot)^\alpha \tau\hat{\varphi} \in L_2(\mathbb{R}^s). \tag{2.11}$$

From this characterization it also follows that f lies in $S(\varphi)$ and $D^\alpha f$ lies in $L_2(\mathbb{R}^s)$ iff the Fourier transform of $D^\alpha f$ equals $(i\cdot)^\alpha \tau\hat{\varphi}$ for some 2π-periodic function τ such that both $\tau\hat{\varphi}$ and $(i\cdot)^\alpha \tau\hat{\varphi}$ lie in $L_2(\mathbb{R}^S)$. This proves that if $f \in S(\varphi)$ satisfies that $D^\alpha f$ lies in $L_2(\mathbb{R}^s)$ then $D^\alpha f$ lies in $S(D^\alpha\varphi)$. In particular, $D^\alpha(S(\varphi) \cap W_2^m(\mathbb{R}^s)) \subset S(D^\alpha\varphi)$.

THEOREM 2.9. *For $\varphi \in W_2^m(\mathbb{R}^s)$ and $k > 0$, $S(\varphi)$ provides simultaneous approximation order $(m, k+m)$ iff $S(D^\alpha\varphi)$ provides approximation order k in the $L_2(\mathbb{R}^s)$-norm to all functions in $D^\alpha W_2^{k+m}(\mathbb{R}^s)$ for each $|\alpha| = m$.*

Proof. As $D^\alpha(S(\varphi) \cap W_2^m(\mathbb{R}^s))$ is a subspace of $S(D^\alpha\varphi)$, it suffices to prove the sufficiency. An analogous proof to that of Theorem 2.2 shows that if $S(D^\alpha\varphi)$ provides approximation order k to all functions in $D^\alpha W_2^{k+m}(\mathbb{R}^s)$ then

$$|\cdot|^{-k} \Lambda_\alpha := |\cdot|^{-k} \left(1 - \frac{(\cdot)^{2\alpha} |\hat{\varphi}|^2}{[\hat{\varphi}, \hat{\varphi}]_\alpha}\right)^{1/2}$$

is essentially bounded on a neighborhood of the origin. By Lemma 2.3, this is equivalent to that

$$\sum_{\beta \neq 0} |\hat{\varphi}(\cdot - 2\pi\beta)|^2 (\cdot - 2\pi\beta)^{2\alpha} \leqslant C_\alpha |\cdot|^{2k} (\cdot)^{2\alpha} |\hat{\varphi}|^2 \tag{2.12}$$

holds a.e. on some neighborhood Ω_α of the origin, with $C_\alpha > 0$ a constant. Multiplying (2.12) by a proper binomial coefficient and summing them up for all $|\alpha| = m$, we then obtain that

$$\sum_{\beta \in \mathbb{Z}^s \setminus 0} |\hat{\varphi}(\cdot - 2\pi\beta)|^2 \, |\cdot - 2\pi\beta|^{2m} \leqslant C \, |\cdot|^{2(k+m)} \, |\hat{\varphi}|^2$$

holds a.e. on the intersection of these finitely many neighborhoods of the origin, with C the maximum of the C_α. From Corollary 2.5 it follows that $S(\varphi)$ provides simultaneous approximation order $(m, k+m)$. ∎

From (2.12) we know that $S(D^\alpha \varphi)$ providing approximation order k to all functions in $D^\alpha w_2^{k+m}(\mathbb{R}^s)$ is equivalent to $S(D^\alpha \varphi)$ providing approximation order k to all functions in $W_2^k(\mathbb{R}^s)$.

3. Approximation by Semidiscrete Convolution

In this section we study approximation via quasi-interpolants of semidiscrete convolution type. By definition, a quasi-interpolant with the generator φ is the linear mapping defined by

$$Q_\lambda : f \mapsto \sum_{\alpha \in \mathbb{Z}^s} \varphi(\cdot - \alpha) \langle \lambda, f(\cdot - \alpha) \rangle$$

for some selected linear functional λ. Denote by

$$\sigma_h : f \mapsto f(\cdot / h)$$

the scaling operator. The well known argument of polynomial reproduction states that if a compactly supported $\varphi \in L_p(\mathbb{R}^s)$ satisfies the Strang-Fix conditions of order k then one can construct a linear functional λ locally dependent on f such that

$$\|\sigma_h Q_\lambda \sigma_{1/h} f - f\|_p = O(h^k), \qquad \forall f \in W_p^k(\mathbb{R}^s). \tag{3.1}$$

When (3.1) holds, Q_λ is said to provide approximation order k. This result has been generalized to noncompactly supported generators that satisfy certain decay conditions at ∞, see the references given in Section 1. Due to the nature of the argument of polynomial reproduction, decay conditions on φ are unavoidable.

In the following we give a new approach for construction of quasi-interpolants that use function values of f on scaled lattices $h\mathbb{Z}^s$ only, for noncompactly supported $\varphi \in W_2^m(\mathbb{R}^s)$. In contrast to the argument of polynomial reproduction that requires Q_λ be equal to the identity mapping on the space of all polynomials of degree $<k$, we choose λ so that Q_λ equals the

identity mapping on a subspace of $S(\varphi)$ that provides the same simultaneous approximation order (m, k), with $m > s/2$. Since $l_2(\mathbb{Z}^s)$ is a Hilbert space and, for $f \in W_2^m(\mathbb{R}^s)$ with $m > s/2$, $\sum_{\alpha \in \mathbb{Z}^s} |f(\alpha)|^2$ is finite, any quasi-interpolant of this type is determined by a sequence in $l_2(\mathbb{Z}^s)$. We shall show how to construct a sequence $b \in l_2(\mathbb{Z}^s)$ such that the scaled semi-discrete convolution mapping

$$\sigma_h Q_b \sigma_{1/h} f := \sum_{\alpha \in \mathbb{Z}^s} \varphi(\cdot/h - \alpha) \sum_{\beta \in \mathbb{Z}^s} b(\alpha - \beta) f(h\beta)$$

achieves the approximation order provided by $S(\varphi)$, when $S(\varphi)$ provides simultaneous approximation order (m, k) with $m \geqslant s/2$ and $1/\tilde{\varphi}$ is essentially bounded on some neighborhood of the origin.

Let $m > s/2$ and $f \in W_2^m(\mathbb{R}^s)$, by the Sobolev lemma, f is also continuous. Denote the h-symbol of f by

$$\widetilde{f_h} := h^s \sum_{\alpha \in \mathbb{Z}^s} f(h\alpha)\, e_\alpha(h\cdot)$$

and the characteristic function of $[-\pi/h..\pi/h)^s$ by χ_h. By Theorem 6 in [7], if f lies in $W_2^m(\mathbb{R}^s)$ then

$$\widetilde{f_h} = \sum_{\alpha \in \mathbb{Z}^s} \hat{f}(\cdot + 2\pi\alpha/h) \tag{3.2}$$

holds a.e. on $\mathbb{R}^s$ and there exists a constant C independent of h such that

$$\|(\chi_h \widetilde{f_h})^\vee - f\|_{j,2} \leqslant C h^{m-j} |f|_{m,2}, \qquad \forall f \in W_2^m(\mathbb{R}^s)$$

for all integers j satisfying that $0 \leqslant j \leqslant m$. In particular, for $j = 0$,

$$\|\chi_h \widetilde{f_h}\|_2 = (2\pi)^{s/2} \|(\chi_h \widetilde{f_h})^\vee\|_{0,2} \leqslant \|f\|_2 + C h^m |f|_{m,2}.$$

Since $\|\chi_h \widetilde{f_h}\|_2 = (2\pi)^{s/2} \|f_h\|_{l_2}$ with $\|f_h\|_{l_2} = (h^s \sum_{\alpha \in \mathbb{Z}^s} |f(h\alpha)|^2)^{1/2}$, thus we obtain

Corollary 3.1. *If $f \in W_2^m(\mathbb{R}^s)$ with $m > s/2$, then $\|f_h\|_{l_2} \leqslant (\|f\| + C h^m |f|_{m,2})$ for some constant C independent of h and f.*

As proved by Jia and Michelli [13], if $\varphi \in L_2(\mathbb{R}^s)$ satisfies

$$\sum_{\alpha \in \mathbb{Z}^s} |\varphi(\cdot - \alpha)| \in L_2([0..1]^s), \tag{3.3}$$

then

$$\varphi^{*\prime} : f \mapsto \sum_{\alpha \in \mathbb{Z}^s} \varphi(\cdot - \alpha) f(\alpha)$$

is a bounded mapping from $l_2(\mathbb{Z}^s)$ to $L_2(\mathbb{R}^s)$. From Corollary 3.1 we know that if φ satisfies (3.3) then $\varphi *'$ maps $W_2^m(\mathbb{R}^s)$ into $L_2(\mathbb{R}^s)$ continuously. For finitely-supported sequence a,

$$\|\varphi *' a\|_2^2 = \int_{\mathbb{R}^s} |\varphi *' a|^2 = (2\pi)^{-s} \int_{\mathbb{R}^s} |\hat{\varphi}\hat{a}|^2$$

with $\hat{a}$ the **Fourier series** of a. Since $\hat{a}$ is 2π-periodic,

$$\|\varphi *' a\|_2^2 = (2\pi)^s \int_{[-\pi..\pi]^s} |\hat{a}|^2 \, [\hat{\varphi}, \hat{\varphi}].$$

It follows from this that $\varphi *' : l_2(\mathbb{Z}^s) \to L_2(\mathbb{R}^s)$ is bounded iff $[\hat{\varphi}, \hat{\varphi}] \in L_\infty([-\pi..\pi]^s)$ [3].

PROPOSITION 3.2. *Assume that $\varphi \in W_2^m(\mathbb{R}^s)$ with $m > s/2$ such that $[\hat{\varphi}, \hat{\varphi}]$ is essentially bounded. Then, $\varphi *' f = f$ for every $f \in S(\varphi) \cap W_2^m(\mathbb{R}^s)$ iff $\hat{\varphi} = \hat{\varphi}\tilde{\varphi}$.*

Proof. For any $a \in l_2(\mathbb{Z}^s)$, let $a_n(\alpha) := a(\alpha)$ for $|\alpha| \leqslant n$ and $a_n(\alpha) := 0$ otherwise. It is clear that $\widehat{a_n} \to \hat{a}$ in $L_2([-\pi..\pi]^s)$. Since we assume that $[\hat{\varphi}, \hat{\varphi}] \in L_\infty([-\pi..\pi]^s)$,

$$\lim_{n \to \infty} \int_{\mathbb{R}^s} |\hat{\varphi}(\widehat{a_n} - \hat{a})|^2 = \lim_{n \to \infty} \int_{[-\pi..\pi]^s} [\hat{\varphi}, \varphi] \, |\widehat{a_n} - \hat{a}|^2 = 0.$$

Therefore, for any $a \in l_2(\mathbb{Z}^s)$, $\widehat{\varphi *' a} = \hat{\varphi}\hat{a}$. In particular,

$$\widehat{\varphi *' f} = \hat{\varphi}\tilde{f}, \qquad \forall f \in W_2^m(\mathbb{R}^s). \tag{3.4}$$

From (3.4) it follows that $\varphi *' \varphi = \varphi$ iff $\hat{\varphi} = \hat{\varphi}\tilde{\varphi}$. For any $f \in S(\varphi) \cap W_2^m(\mathbb{R}^s)$, then there exists a 2π-periodic function τ such that $\hat{f} = \tau\hat{\varphi}$. From this and (3.2) it follows that $\tilde{f} = \tau\tilde{\varphi}$. Multiplying $\tilde{f} = \tau\tilde{\varphi}$ by $\hat{\varphi}$ yields that $\hat{\varphi}\tilde{f} = \tau\hat{\varphi}\tilde{\varphi} = \tau\hat{\varphi} = \hat{f}$. Therefore, $\varphi *' f = f$ follows from (3.4). ∎

When $\tilde{\varphi}$ is continuous and $\tilde{\varphi} \neq 0$ on $\mathbb{R}^s$, it is clear that

$$\hat{\psi} := \frac{\hat{\varphi}}{\tilde{\varphi}}$$

belongs to $W_2^m(\mathbb{R}^s)$ and $\tilde{\psi} = 1$. Thus $\psi \in S(\varphi)$ and $\varphi \in S(\psi)$ because $\hat{\varphi} = \tilde{\varphi}\hat{\psi}$. Therefore, $S(\varphi) = S(\psi)$. As we know, $\hat{f}(x)$ goes to 0 as $|x| \to \infty$ if f lies in $L_1(\mathbb{R}^s)$. When φ lies in $W_1^m(\mathbb{R}^s)$, we have that

$$\lim_{|x| \to \infty} |x|^m \, |\hat{\varphi}(x)| = 0.$$

Therefore, there is a constant C such that $|\hat{\varphi}(x)| \leqslant C|x|^{-m}$ for all x, noticing that $\hat{\varphi}$ is continuous. Thus, for $\varphi \in W_1^m(\mathbb{R}^s)$ with $m > s/2$, from (3.2) we know that $\tilde{\varphi}$ is continuous. It is clear that $\tilde{\varphi}$ is continuous when $\sum_{\alpha \in \mathbb{Z}^s} |\varphi(\alpha)|$ is finite.

For any $f \in S(\varphi) \cap W_2^m(\mathbb{R}^s)$, $\sigma_h(\varphi^{*\prime})\, \sigma_{1/h}(\sigma_h f) = \sigma_h(\varphi^{*\prime} f)$. Therefore, $\varphi^{*\prime}$ is equal to the identity mapping on $S(\varphi) \cap W_2^m(\mathbb{R}^s)$ implies $\sigma_h(\varphi^{*\prime})\, \sigma_{1/h}$ equal to the identity mapping on $(\sigma_h S(\varphi)) \cap W_2^m(\mathbb{R}^s)$.

THEOREM 3.3. *Let $m > s/2$ and $\varphi \in W_2^m(\mathbb{R}^s)$ such that $\hat{\varphi} = \hat{\varphi}\tilde{\varphi}$ and $[\hat{\varphi}, \hat{\varphi}]$ is essentially bounded. If $S(\varphi)$ provides simultaneous approximation order (m, k), then there exists a constant C independent of h such that*

$$\|\sigma_h(\varphi^{*\prime})\, \sigma_{1/h} f - f\|_2 \leqslant Ch^k(\|f\|_{m,2} + \|f\|_{k,2}), \qquad \forall f \in W_2^m(\mathbb{R}^s) \cap W_2^k(\mathbb{R}^s).$$

Proof. As we know from Theorem 3.2, the condition that $\hat{\varphi} = \hat{\varphi}\tilde{\varphi}$ together with the boundedness of $[\hat{\varphi}, \hat{\varphi}]$ implies that $\varphi^{*\prime}$ coincides with the identity mapping on $S(\varphi) \cap W_2^m(\mathbb{R}^s)$. As assumed, we have $g \in S(\varphi)$ such that

$$h^{-2k} \|f - g^h\|_2^2 + h^{-2(k-m)} |f - g^h|_{m,2}^2 \leqslant C^2(\|f\|_{m,2} + \|f\|_{k,2})^2 \quad (3.5)$$

for some constant C independent of h and f, where $g^h := g(\cdot/h)$. Since $g^h \in \sigma_h(S(\varphi)) \cap W_2^m(\mathbb{R}^s)$, $\sigma_h(\varphi^{*\prime})\, \sigma_{1/h} g^h = g^h$. Therefore,

$$\begin{aligned}\|\sigma_h(\varphi^{*\prime})\, \sigma_{1/h} f - g^h\|_2^2 &= \|\sigma_h(\varphi^{*\prime})\, \sigma_{1/h}(f - g^h)\|_2^2 \\ &= h^s \int_{\mathbb{R}^s} |\varphi^{*\prime} \sigma_{1/h}(f - g^h)|^2 \leqslant C_\varphi \|(f - g^h)_h\|_{l_2}^2,\end{aligned}$$

with $C_\varphi := \|[\hat{\varphi}, \hat{\varphi}]\|_{L_\infty([-\pi..\pi]^s)}$. Using Corollary 3.1, we obtain

$$\|\sigma_h(\varphi^{*\prime})\, \sigma_{1/h} f - g^h\|_2 \leqslant C_0(\|f - g^h\|_2 + h^m |f - g^h|_{m,2})$$

for some constant C_0 independent of h and f. This together with (3.5) implies that

$$\|\sigma_h(\varphi^{*\prime})\, \sigma_{1/h} f - g^h\|_2 \leqslant 2C_0\, Ch^k(\|f\|_{m,2} + \|f\|_{k,2}).$$

Thus we obtain

$$\begin{aligned}\|\sigma_h(\varphi^{*\prime})\, \sigma_{1/h} f - f\|_2 &\leqslant \|f - g^h\|_2 + \|\sigma_h(\varphi^{*\prime})\, \sigma_{1/h} f - g^h\|_2 \\ &\leqslant (1 + 2C_0 C)\, h^k(\|f\|_{m,2} + \|f\|_{k,2}). \quad \blacksquare\end{aligned}$$

We next consider the case where the condition that $\hat{\varphi} = \hat{\varphi}\tilde{\varphi}$ fails to hold but there exists a neighborhood $B \subset [-\pi..\pi]^s$ of the origin such that $1/\tilde{\varphi}$ is essentially bounded on B. As we saw in the above proof, it is sufficient to assume that $\varphi^{*\prime}$ coincides with the identity mapping on any subspace of $S(\varphi) \cap W_2^m(\mathbb{R}^s$ whose closure in $L_2(\mathbb{R}^s)$ provides the same simultaneous approximation order as $S(\varphi)$ does. This motivates the following approach.

Let χ_B be the characteristic function of B and τ_B the 2π-periodic function defined by

$$\tau_B := \frac{\chi_B}{\tilde{\varphi}}, \qquad \text{on } [-\pi..\pi]^s.$$

Note that $\tau_B = 0$ on $[-\pi..\pi]^s \backslash B$ according to our convention. Consider the function ψ which is determined by

$$\hat{\psi} := \tau_B \hat{\varphi}. \tag{3.6}$$

It follows that $\psi \in S(\varphi)$. Hence, $S(\psi) \subset S(\varphi)$. Since τ_B is 2π-periodic and essentially bounded, $\psi \in W_2^m(\mathbb{R}^s)$, and

$$[\hat{\psi}, \hat{\psi}] = |\tau_B|^2 [\hat{\varphi}, \hat{\varphi}], \tag{3.7}$$

it follows that $[\hat{\psi}, \hat{\psi}]$ is essentially bounded as long as $[\hat{\varphi}, \hat{\varphi}]$ is. By (3.2), $\tilde{\psi} = \chi_B$ on $[-\pi..\pi]^s$. Therefore,

$$\hat{\psi} = \tau_B \hat{\varphi} = \hat{\psi}\tilde{\psi} \tag{3.8}$$

Moreover, on $[-\pi..\pi]^s$ we have

$$\sum_{\alpha \in \mathbb{Z}^s \backslash 0} |\cdot - 2\pi\alpha|^{2m} |\hat{\psi}(\cdot - 2\pi\alpha)|^2 = |\tau_B|^2 \sum_{\alpha \in \mathbb{Z}^s \backslash 0} |\cdot - 2\pi\alpha|^{2m} |\hat{\varphi}(\cdot - 2\pi\alpha)|^2.$$

From Theorem 2.5 it follows that the subspace $S(\psi)$ provides the same simultaneous approximation order as $S(\varphi)$ does. Since τ_B is a measurable and bounded 2π-periodic function, $\tau_B = \hat{b} = \sum_{\alpha \in \mathbb{Z}^s} b(\alpha)\, e_\alpha$ for some $b \in l_2(\mathbb{Z}^s)$ and

$$b^*: l_2(\mathbb{Z}^s) \to l_2(\mathbb{Z}^s): a \mapsto b^* a := \sum_{\alpha \in \mathbb{Z}^s} b(\cdot - \alpha)\, a(\alpha)$$

is a bounded linear mapping. Hence, the linear functional

$$\lambda_B: a \in l_2(\mathbb{Z}^s) \mapsto b^* a(0)$$

is continuous and $\langle \lambda_B, a(\cdot + \beta)\rangle = (b^* a)(\beta)$ for each $\beta \in \mathbb{Z}^s$. Call a quasi-interpolant

$$Q_\lambda : f \mapsto \sum_{\alpha \in \mathbb{Z}^s} \varphi(\cdot - \alpha)\langle \lambda, f(\cdot - \alpha)\rangle$$

a **controlled approximant** (with respect to φ) if, for $h > 0$,

$$h^s \sum_{\alpha \in \mathbb{Z}^s} |\langle \lambda, f(h(\cdot - \alpha))\rangle|^2 \leqslant C \, \|f\|^2_{m,2}$$

for some constant C independent of h and f.

THEOREM 3.4. *Assume that $\varphi \in W_2^m(\mathbb{R}^s)$ with $m > s/2$ such that $[\hat{\varphi}, \hat{\varphi}]$ is essentially bounded and there exists a neighborhood $B \subset [-\pi..\pi]^s$ of the origin on which $1/\tilde{\varphi}$ is essentially bounded. If $S(\varphi)$ provides simultaneous approximation order (m, k), then $(\varphi^{*\prime} b)^{*\prime}$ is a controlled approximant that provides approximation order k, where b consists the Fourier series coefficients of $\chi_B/\tilde{\varphi}$.*

Proof. Let $\psi := \varphi^{*\prime} b$, then $\hat{\psi} = \tau_B \hat{\varphi}$. It follows from (3.8) that $\hat{\psi} = \hat{\psi}\tilde{\psi}$. The assumption and (3.7) show that $[\hat{\psi}, \hat{\psi}]$ is essentially bounded. As shown above, $S(\psi)$ and $S(\varphi)$ provide the same simultaneous approximation order. By Theorem 3.3, $\psi^{*\prime} = (\varphi^{*\prime} b)^{*\prime}$ achieves the approximation order provided by $S(\varphi)$.

We next prove that it is a controlled approximant. Since τ_B is essentially bounded, by Corollary 3.1,

$$\begin{aligned}
h^s \sum_{\beta \in \mathbb{Z}^s} \left| \sum_{\alpha \in \mathbb{Z}^s} b(\alpha) f(h(\alpha - \beta)) \right|^2 &= (2\pi)^{-s} \int_{[-\pi..\pi]^s} |\tau_B|^2 \, |\widetilde{f_h}|^2 \\
&\leqslant (2\pi)^{-s} \, \|\tau_B\|^2_{L_\infty([-\pi..\pi]^s)} \int_{[-\pi..\pi]^s} |\widetilde{f_h}|^2 \\
&= \|\tau_B\|^2_{L_\infty([-\pi..\pi]^s)} \, \|f_h\|^2_{l_2} \\
&\leqslant C \, \|\tau_B\|^2_{L_\infty([-\pi..\pi]^s)} \, \|f\|^2_{m,2},
\end{aligned}$$

for some constant C independent of $h < 1$ and f. This completes the proof. ∎

When φ satisfies $\sum_{\alpha \in \mathbb{Z}^s} |\varphi(\alpha)| < \infty$, $\tilde{\varphi}$ is continuous. So the following corollary is immediate.

COROLLARY 3.5. *Assume that $\varphi \in W_2^m(\mathbb{R}^s)$ with $m > s/2$ such that $[\hat{\varphi}, \hat{\varphi}]$ is essentially bounded, $\tilde{\varphi}(0) \neq 0$, and $\sum_{\alpha \in \mathbb{Z}^s} |\varphi(\alpha)| < \infty$. If $S(\varphi)$ provides simultaneous approximation order (m, k) then there exists a neighborhood B*

of the origin such that the scaled semi-discrete convolution mapping with ψ achieves the approximation order k of $S(\varphi)$ and provides a controlled approximation with respect to φ, where ψ is determined by (3.6).

It is well known that if a controlled approximant Q_λ with compactly supported φ provides approximation order $k>0$ then φ satisfies the Strang–Fix conditions of order k [16]. Corollary 3.5 can be applied to many cases where φ satisfies certain decay condition. For instance, when $\varphi \in W_2^m(\mathbb{R}^s)$ satisfies

$$\sum_{\alpha \in \mathbb{Z}^s} |\cdot - \alpha|^k \, |D^\beta \varphi(\cdot - \alpha)| \in L_2([0..1]^s), \qquad \forall |\beta| \leqslant m \tag{3.9}$$

with $k \geqslant m$, as proved by [17], if φ satisfies the Strang–Fix conditions of order k then $S(\varphi)$ provides simultaneous approximation order (m, k). One can verify that (3.9) implies that φ lies in $W_1^m(\mathbb{R}^s)$. Therefore (3.9) infers that $\hat{\varphi}$ is continuous when $m > s/2$. When φ is compactly supported, $\tilde{\varphi}$ is a trigonometric polynomial. Thus we have

COROLLARY 3.6. *Let $\varphi \in W_2^m(\mathbb{R}^s)$, with m an integer $> s/2$, be compactly supported. If φ satisfies the Strang-Fix conditions of order k, then there exists $b \in l_2(\mathbb{Z}^s)$ such that the semi-discrete convolution $(\varphi *' b) *'$ has approximation order k.*

In the theory of cardinal interpolation, it is well known that, for a continuous and compactly supported function φ, if $\tilde{\varphi}$ does not vanish then $\psi *' f = f$ on $\mathbb{Z}^s$ for all bounded continuous functions f, where ψ is determined by $\hat{\psi} = \hat{\varphi}/\tilde{\varphi}$. Particularly, it follows that $\psi *'$ provides approximation order k when φ satisfies the Strang–Fix conditions of order k. So, the last corollary is a generalization of this result. When φ is a box spline, it is well known that, in general, $\tilde{\varphi}$ vanishes somewhere on $[-\pi..\pi]^s$, see [5] and [9] for the definition of box splines. Since φ now is compactly supported and $\tilde{\varphi}(0) > 0$, it follows from the last corollary that there exists $b \in l_2(\mathbb{R}^s)$ such that $(\varphi *' b) *'$ achieves the approximation order provided by $S(\varphi)$.

REFERENCES

1. C. DE BOOR, Quasiinterpolants and approximation power of multivariate splines, *in* "Computation of Curves and Surfaces" (M. Gasca and C. Michelli, Eds.), pp. 313–345, Kluwer, Dordrecht, 1990.
2. C. DE BOOR, The polynomials in the linear span of integer translates of a compactly supported function, *Constr. Approx.* **3** (1987), 199–208.
3. C. DE BOOR, R. DEVORE, AND A. RON, Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^s)$, *Trans. Amer. Math. Soc.* **341** (1994), 787–806.

4. C. de Boor, R. DeVore, and A. Ron, The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^s)$, *J. Funct. Anal.* **119** (1994), 37–78.
5. C. de Boor, K. Höllig, and S. Riemenschneider, "Box Splines," Springer-Verlag, New York, 1993.
6. C. de Boor and R. Q. Jia, A sharp upper bound on the approximation order of smooth bivariate *pp* functions, *J. Approx. Theory* **72** (1993), 37–78.
7. J. Bramble and S. Hilbert, Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and spline interpolation, *SIAM J. Numer. Anal.* **7** (1970), 112–124.
8. E. Cheney and W. Light, Quasi-interpolation with translates of a function having non-compact support, *Constr. Approx.* **1** (1992), 35–48.
9. C. Chui, "Multivariate Splines," CBMS-NSF Series Appl. Math., Vol. 54, SIAM Publications, Philadelphia, PA, 1988.
10. C. Chui, K. Jetter, and J. Ward, Cardinal interpolation with multivariate splines, *Math. Comp.* **48** (1987), 711–724.
11. R. Q. Jia, The Topelitz theorem and its applications to approximation theory and linear PDE's, *Trans. Amer. Math. Soc.*, to appear.
12. R. Q. Jia and J. Lei, Approximation by multiinteger translates of functions having non-compact support, *J. Approx. Theory* **68** (1992).
13. R. Q. Jia and C. A. Micchelli, using the refinement equations for the construction of pre-wavelets. II. Powers of two, *in* "Curves and Surfaces" (P. J. Laurent, A. Le Méhauté, and L. Schumaker, Eds.), pp. 209–246, Academic Press, New York/London, 1991.
14. A. Ron, A characterization of the approximation order of multivariate splines spaces, *Studia Math.* **98** (1991), 73–90.
15. I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions, Parts A and B, *Quart. Appl. Math.* **4** (1946), 45–99, 112–141.
16. G. Strang and G. Fix, A Fourier analysis of the finite element variational method, *in* "C.I.M.E.II Ciclo 1971, Constructive Aspects of Functional Analysis" (G. Geymonat, Ed.), pp. 793–840, 1973.
17. K. Zhao, Simultaneous approximation from PSI spaces, *J. Approx. Theory* **81** (1995), 166–184.
18. K. Zhao, Approximation from locally finite-dimensional shift-invariant spaces, *Proc. Amer. Math. Soc.*, in press.